Complex Integration

As we have mentioned in the previous chapter, one of the central results of this course is the fact that holomorphic functions are in fact complex-analytic. To prove this we need to develop the machinery of complex integration. The main result is—just as in the case of real analysis—the link between integration and differentiation. We will prove a version of the fundamental theorem of calculus for complex line integrals.

1 Complex line integrals

Let $f : [a, b] \to \mathbb{C}$ be continuous and let $f = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$

Let $\alpha, \beta \in \mathbb{C}$ and let $g : [a, b] \to \mathbb{C}$, then

$$\int_a^b (\alpha f + \beta g)dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt.$$

We have the following useful inequality:

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt.$$

To see this, let $\int_a^b f(t)dt = re^{i\theta}$. Then

$$\begin{aligned} \left| \int_a^b f(t)dt \right| &= r = e^{-i\theta} \int_a^b f(t)dt \\ &= \mathrm{Re}\left[e^{-i\theta} \int_a^b f(t)dt \right] = \int_a^b \mathrm{Re}\left[e^{-i\theta} f(t) \right] dt \\ &\leq \int_a^b |f(t)|dt. \end{aligned}$$

Definition 1. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve and let $f : \gamma^* \to \mathbb{C}$ be continuous. We define

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Note that the above integral is the limit of the complex Riemann sums

$$\sum_i f(z_i)(z_{i+1} - z_i),$$

where the points $z_i := \gamma(t_i)$ are the vertices of a polygonal approximation of γ. Replacing $z_{i+1} - z_i$ by $|z_{i+1} - z_i|$ above, we recover the definition of integration with respect to arc-length

$$\int_\gamma f(z)|dz| := \int_\gamma f(z)ds.$$

It is clear that $\int_\gamma dz = \gamma(b) - \gamma(a)$.

Now, let $\Gamma = \gamma \circ \Phi$ where $\Phi : [c, d] \to [a, b]$ is a strictly increasing homeomorphism, i.e., orientation preserving. Then

$$\int_\Gamma f(z)dz = \int_c^d f(\Gamma(u))\Gamma'(u)du.$$

By chain rule, this is same as

$$\int_c^d f(\gamma \circ \Phi(u))\gamma'(\Phi(u))\Phi'(u)du.$$

Setting $t = \Phi(u)$ and applying change of variables, we see that the above is same as

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_\gamma f(z)dz.$$

If Φ were orientation reversing then $\int_\gamma f(z)dz = -\int_\Gamma f(z)dz$.

Now suppose γ is a path. Then, we can write $\gamma = \gamma_1 + \cdots + \gamma_n$ where γ_i's are all C^1 curves such that the ending point of γ_i coincides with the starting point of γ_{i+1}. Then we define

$$\int_\gamma f(z)dz := \sum_i \int_{\gamma_i} f(z)dz.$$

We have the following extremely useful inequality:

$$\left|\int_\gamma f(z)dz\right| \le \int_a^b |f(\gamma(t))||\gamma'(t)|dt = \int_\gamma |f(z)|dz.$$

The use of this inequality is easily seen when f is bounded above by M. Then, we immediately see that

$$\left|\int_\gamma f(z)dz\right| \le ML(\gamma)$$

2 The Fundamental Theorem of Complex Calculus

Definition 2. Let $f : U \to \mathbb{C}$ be continuous. We say that $F : U \to \mathbb{C}$ is a *primitive* or *anti-derivative* of f if $F \in \mathscr{O}(U)$ and $F' \equiv f$.

Theorem 3 (The fundamental theorem of complex calculus I)**.** *Let $f : U \to \mathbb{C}$ have a primite F. Let $\gamma : [a, b] \to \mathbb{C}$ be a contour. Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if $\gamma(b) = \gamma(a)$ (i.e., γ is a closed contour) then $\int_\gamma f(z)dz = 0$.

Proof. By definition

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))dt = F(\gamma(b)) = F(\gamma(a)),$$

where we have used the chain rule followed by the fudamental theorem of calculus applied to the real and imaginary parts of the integral. □

Another version of the fundamental theorem of calculus says that any continuous function $f : [a,b] \to \mathbb{R}$ has an anti-derivative. This is *not* true for complex valued functions. In fact, it is not even true that holomorphic functions have a primitive. The existence (or rather the non-existence) of primitives in a domain is intimately tied together with topology of the domain. The following question is the second central question of the course:

Under what conditions on a domain U is it true that any holomorphic function on U has a primitive?

Theorem 4 (Fundamental theorem of complex calculus II). *Let $U \subset \mathbb{C}$ be a domain and let $f : U \to \mathbb{C}$ be a continuous function such that $\int_\gamma f dz = 0$ for all closed contours $\gamma : [a.b] \to U$. Then f has a primitive.*

Proof. Fix $z_0 \in U$. Define

$$F(z) := \int_\gamma f(w)dw,$$

where $\gamma : [a,b] \to U$ is any contour with $\gamma(a) = z_0$ and $\gamma(b) = z$. By our hypothesis on f, the above integral is well-defined and does not depend on our choice of contour. To show differentiability of F at z, observe that we can write the difference quotient $\frac{F(z+h)-F(z)}{h}$ as

$$\int_{\gamma_{z\to z+h}} f(w)dw,$$

where $\gamma_{z\to z+h}$ is the straight line path joinig z_0 and z. If h is a real number, it follows that

$$\begin{aligned}\int_{\gamma_{z\to z+h}} f(w)dw - f(z) &= \int_{\gamma_{z\to z+h}} f(z)dz - \frac{1}{h}\int_{\gamma_{z\to z+h}} f(z)dw \\ &= \frac{\int_{\gamma_{z\to z+h}} (f(w) - f(z))dw}{h}.\end{aligned}$$

As $h \to 0$, the ML-inequality and the continuity of f shows that the above limit goes to 0. We have shown $\frac{\partial F}{\partial x} = f$. Now assume h is real and consider

$$\frac{\partial F}{\partial y} = \frac{F(z+ih) - F(z)}{h}.$$

We get

$$\begin{aligned}\int_{\gamma_{z\to z+ih}} f(w)dw - if(z) &= \int_{\gamma_{z\to z+ih}} f(z)dz - \frac{1}{h}\int_{\gamma_{z\to z+ih}} f(z)dw \\ &= \frac{\int_{\gamma_{z\to z+ih}} (f(w) - f(z))dw}{h} \to 0.\end{aligned}$$

This shows that $\frac{\partial F}{\partial y} = i\frac{\partial F}{\partial y}$ whence F is holomorphic and $F' = f$.

□

The above version of the fundamental theorem of calculus leads us to reformulate the second central question as follows:

Under what conditions on a domain U is it true that for any holomorphic function on U, the integral over a closed contour is zero?

The second central question will be answered by the general Cauchy theorem to be proved later.

3 The Cauchy–Goursat Theorem

Theorem 5 (Goursat). *Let $R = [a, b] \times [c, d]$ be a rectangle in U. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then*

$$\int_{\partial R} f(z)dz = 0.$$

Here the ∂R is traversed anti-clockwise.

Proof. Assume the integral is non-zero and let m be its absolute value. We divide R into 4 equal smaller rectangles and we give the boundaries the positive orientation. The integral of f on at least one of this smaller rectangles has absolute value $\geq m/4$. Iterating this process, we construct a sequence of rectangles R_n with $R_0 = R, R_{n+1} \subset R_n$ and

$$\left|\int_{\partial R_n} f(z)dz\right| \geq \frac{m}{4^m}.$$

The intersection of the R_n's is a single point a. Take $\varepsilon > 0$. As f is $\mathbb{C}$-differentiable at a, we can find a neighbourhood V of a in U for which

$$|f(z) - f(a) - (z - a)f'(a)| \leq \varepsilon|z - a| \; \forall z \in V.$$

Choose N large so that $R_N \subset V$. Then

$$\left|\int_{\partial R_N} (f(z) - f(a) - (z - a)f'(a))dz\right| \leq \varepsilon \left(\frac{\delta}{2^N}\right)^2,$$

where $\delta :=$ the perimeter of R and thus $\delta/2^N$ is the perimeter of R_N. Now, a simple direct computation shows that

$$\int_{\partial R_N} f(a) + (z - a)f'(a)dz = 0.$$

This shows that

$$\left|\int_{\partial R_N} f(z)dz\right| \leq \varepsilon\frac{\delta^2}{4^n}.$$

If $\varepsilon < \frac{m}{\delta^2}$, then we get a contradiction. ◻

Corollary 6. *The Cauchy–Goursat theorem remains true if we assume that f is continuous on U and holomorphic outside a finite subset $S \subset U$.*

Proof. Let M be an upper bound of $|f|$ on U. As in the proof of the Cauchy–Goursat theorem, subdivide R into 4^m sub-rectangles each of which is 2^n times smaller than R. Each point of S can belong to at most 4 sub-rectangles. Thus, by the Cauchy–Goursat theorem, the integral of f on all but $4 \times |S|$ sub-rectangles is 0. Hence

$$\left| \int_{\partial R} f(z)dz \right| \leq 4 \times |S| M \frac{\delta}{2^n}.$$

This integral clearly goes to 0 as $n \to \infty$. □

Theorem 7. *Any holomorphic function in an open set $U \subset \mathbb{C}$ is automatically analytic in U.*

Proof. Let $a \in U$. Let R be a rectangle in U such that $a \in \mathring{R}$. Let $r > 0$ be such that $D(a,r) \subset R$ and let $b \in D(a,r)$. Let $g : U \to \mathbb{C}$ be the function defined by

$$g(z) = \begin{cases} \frac{f(z)-f(b)}{z-b} & z \neq b \\ f'(z) & z = 0. \end{cases}$$

Clearly, $g \in \mathscr{O}(U \setminus \{b\}) \cap C(U)$. By the previous corollary $\int_{\partial R} g(z)dz = 0$. Hence

$$\int_{\partial R} \frac{f(z)}{z-b} dz - f(b) 2\pi i = 0,$$

and thus

$$f(b) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-b} dz.$$

For any $z \in \partial R$, $\frac{|b-a|}{|z-a|} \leq \frac{|b-a|}{r} < 1$. Thus

$$\sum_{n=0}^{\infty} \frac{(b-a)^n}{(z-a)^{n+1}} = \frac{1}{z-a} \cdot \frac{1}{1 - \frac{b-a}{z-a}} = \frac{1}{z-b},$$

and this convergence is uniform for $z \in R$. Thus

$$f(b) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-b} = \sum_{n=0}^{\infty} (b-a)^n \int_{\partial P} \frac{f(z)}{(z-a)^{n+1}} \frac{dz}{2\pi i}.$$

This proves that f can be expanded in a power series near a proving that f is analytic. □

3.1 Morera's theorem

Theorem 8. *Let $f : U \to \mathbb{C}$ be a continuous function such that for each rectangle $R \subset U$, we have $\int_{\partial R} f(z)dz = 0$. Then f is analytic on U.*

Proof. There is no harm in assuming that U is an open disk centred at a. For each $z \in U$, let $F(z)$ be the integral of f along a path from a to z composed of a horizontal segment followed by a vertical segment. By hypothesis, $F(z)$ can also be obtained by integrating f along the path that first consists of a vertical segment and then a horizontal segment. The proof that F is a primitive of f now follows along the same as lines as the proof the second form of the fundamental theorem of calculus and is left to the reader. □

3.2 Cauchy's integral formula for disk and applications

Theorem 9. *Let $f : U \to \mathbb{C}$ be holomorphic and $\overline{D}(a,r) \subset U$. For $b \in D(a,r)$, we have*

$$f(b) = \int_{C(a,r)} \frac{f(z)}{z-b}\frac{dz}{2\pi i},$$

where $C(a,r) = \partial D(a,r)$ with positive orientation. More generally,

$$\frac{f^{(n)}(b)}{n!} = \int_{C(a,r)} \frac{f(z)}{(z-b)^{n+1}}\frac{dz}{2\pi i}$$

Furthermore, the Taylor series expansion of f around a converges unifomly on compact subsets of $D(a,r)$ to f.

Proof. The proof is exactly same as that for the rectangle. □

4 The principle of analytic continuation

We will now study some of the aspects of holomorphic functions that arise out of complex-analyticity. One main distinguishing feature of analytic functions as opposed to C^∞ functions is that analytic functions are more rigid. This is made precise in this section. We first define the order of a holomorphic function at a point.

Lemma 10. *Let $U \subset \mathbb{C}$ be a domain, $a \in U$ and let $f \in \mathscr{O}(U)$. Then the following conditions are all equivalent:*

1. *$f^{(n)}(a) = 0\ \forall n$.*
2. *$f(z) = 0$ in a neighborhood of a.*
3. *$f \equiv 0$ on U.*

Proof. Obviously $(c) \implies (b), (b) \implies a$. It suffices to prove that $(a) \implies (c)$. To do this we use a connectedness argument. Let

$$A := \{z \in U : f(z) = 0 \text{ in some neighborhood of z}\}.$$

Clearly, A is an open set. Let $z_m \in A$ be such that $z_m \to z \in U$. Then $f^{(n)}(z_m) = 0$ by the definition of A and consequently by passing to limits, $f^{(n)}(z) = 0$. But this means that the Taylor expansion of f centered at 0 has all coefficients 0 and this means that f vanishes on some disk around z and hence $z \in A$. This proves that A is closed in U and consequently $A = U$ proving that $(a) \implies (c)$. □

Definition 11. A *discrete* subset $E \subset U$ is a closed subset of U such that each point of E is an isolated point.

Theorem 12 (The principle of analytic continuation)**.** *If $f \in \mathscr{O}(U)$ then either the zeros set of f*

$$Z(f) := \{z \in U : f(z) = 0\}$$

is discrete or f is constant.

Proof. The set $Z(f)$ is clearly closed in U. Let $a \in Z(f)$. Assume that f is non-constant. Then, by the previous lemma, the Taylor series of f around a is of the following form:

$$f(z) = \sum_{n=k}^{\infty} c_n (z-a)^n,$$

where $k > 0$ and $c_k \neq 0$. It is straightforward to see that the power series $\sum_{n=0}^{\infty} c_{n+k}(z-a)^n$ converges in a disk around 0. Moreover, as $c_k \neq 0$, it follows that the function g defined by this series is nowhere zero in a small disk around a. As $f(z) = (z-a)^k g(z)$ in a small disk around a, it follows that a is an isolated point of $Z(f)$. □

5 The maximum principle and open mapping theorem

Recall that the Laplacian $\triangle u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ for $u \in C^2(U)$.

Lemma 13. *If $u : U \to \mathbb{R}$ is C^2-smooth. Suppose $\triangle u \geq 0$ on U. Let $G \Subset U$ be a open (not necessarily connected) set. Then*

$$\sup_{w \in G} u(w) \leq \sup_{w \in \partial G} u(w).$$

Remark 14. The notation $G \Subset U$ (read G is relatively compact in U) means that $\overline{G} \subset U$ and $\overline{G}$ is compact. Here the closure is taken in $\mathbb{C}$.

Proof. First assume that $\triangle u > 0$ on $\overline{G}$. The if $z_0 \in G$ such that z_0 is a point of maximum for u in $\overline{G}$. Then

$$\triangle u(z_0) \leq 0,$$

which is not possible. The lemma is prove with the additional hypothesis that $\triangle u > 0$ on $\overline{G}$. If not, consider the function

$$\triangle u_\varepsilon := u + \varepsilon |z|^2.$$

Then $\triangle u_\varepsilon > \varepsilon$ and therefore from the previous argument

$$\sup_{w \in G} u_\varepsilon(w) \leq \sup_{w \in \partial G} u_\varepsilon(w).$$

The lemma follows by letting $\varepsilon \to 0$. □

Theorem 15 (Maximum principle weak form)**.** *Let $f : U \to \mathbb{C}$ is holomorphic and $G \Subset U$. Then*

$$\sup_{w \in G} |f(w)| \leq \sup_{w \in \partial G} |f(w)|.$$

Proof. Just compute $\triangle |f|^2$. □

Theorem 16 (The open mapping theorem)**.** *Let $f \in \mathscr{O}(U)$. Then either f is a constant or else it is an open map.*

Proof. Let $a \in U$. We may assume that $f(a) = 0$ and show that for small enough $r > 0$ that $f(D(a,r))$ has 0 as an interior point unless f is constant. If f is non-constant, then for some small disk $D(a,r)$ we have $f \neq 0$ on $D^*(a,r)$ (by principle of analytic continuation). Let δ be the minimum value of $|f|$ on the circle of radius r centered at a. Suppose $w \notin f(\overline{D(a,r)})$, then we claim that $|w| \geq 1/2\delta$. We may assume that $|w| < \delta$. Then the function

$$g(z) = \frac{1}{f(z) - w}$$

is holomorphic on $D(a,r)$ and satisfies

$$\frac{1}{|w|} = |g(a)| \leq \sup z \in \partial D(a,r)|g(z)| \leq \frac{1}{\delta - |w|}.$$

This means that $|w| \geq 1/2\delta$ as claimed and we are done. □

Theorem 17 (The maximum principle strong form)**.** *Let $f \in \mathscr{O}(U)$ where U is a bounded domain. Let*

$$M := \{\limsup_{z \to \zeta} |f(z)| : \zeta \in \partial U\}.$$

Then either f is constant or $|f| < M$.

Proof. We may assme that f is an open map. It suffices to show that $|f| \leq M + \varepsilon$ for each $\varepsilon > 0$. This would show that $|f| \leq M$ and we get the desired inequality that $|f| < M$ because f is an open map. For each $\zeta \in \partial U$, using the definition of M, we can find a disk D_ζ such that $|f| < M + \varepsilon$ on $D_\zeta \cap U$. Call the union of these disks D. Then clearly $K := U \setminus D$ is a compact set. Let $G \subset U$ be an open set with $K \subset G$. Then $\partial G \subset D \cap U$. The maximum principle gives us that $|f| \leq M + \varepsilon$ on G and hence on U. The result follows. □